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From square bamboos to Superformula

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Abstract

Improved quantitative methods for forest management, yields and carbon fixation are needed for bamboo. We report here a novel scientific method for quantifying natural forms. Remarkably, the origin of this breakthrough lies in the observation that square bamboos (species of the genus *Chimonobambusa*) are superelliptic. Superellipses can be considered as the simplest extensions of ellipses (and conic sections in general). The generalization of superellipses to the superformula (also known as Gielis curves and transformations GT) extends the scope of a unified geometric description to many more natural shapes, from very small to very large. In the last ten years, superellipses and GT have been successfully tested on over 40,000 biological specimens, including bamboo parts, tree rings, plant leaves, diatoms, bird eggs and starfish. These curves prove to be an excellent scientific tool for the study of natural forms, especially for a precise quantitative description of bamboo cells, fibers, culms and rhizomes, meristems, seeds, and leaves. For example, descriptive terms for leaves such as linear, lanceolate, elliptic are now given a precise quantitative meaning in just a few numbers determined by the simplified Gielis equation.

Keywords: Superellipse, quantification, leaves, culms

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1. Introduction

In the last two decades we have seen an enormous increase in quantitative data in agriculture, horticulture, and forestry, where accurate data on growth and yield are crucial. Methodologies such as remote sensing, image recognition, hyperspectral imaging, machine learning, and artificial intelligence, will become very important in all fields, including forestry and agroforestry. To advance bamboo as the material of choice, these technologies will also need to be used in bamboo. At present, quantification of yield is restricted mainly to the use of allometric equations to estimate biomass storage potential or yields (see e.g. Wang et al, 2021; Abebe et al, 2023; Camargo et al, 2023; Yen, 2023). In taxonomy and morphology in bamboo, qualitative terms and terminology are still the rule.

Allometric equations are essentially power laws and a generalization of parabolas (Gielis, 2023). Our research focused on a different generalization of conic sections (of circles and ellipses) for precise quantification of leaves, culms, meristems of bamboo and a range of leaves, fruits, stomata in other plants. Historically, this development began with the study of *Chimonobambusa quadrangularis*, the square bamboo, and hollow culms of bamboo.

To model these forms, superellipses, and supercircles proved to be excellent models for square bamboos, especially for the cross-sectional forms of the square bamboo, *Chimonobambusa quadrangularis.* When McGowan drew the attention of Western botanists to the square bamboos, which have been known in China and Japan for thousands of years, it caused quite a stir. In 1885 Dyer wrote in Nature: "*The cylindrical form of the stems of grasses is so general a feature in the family that the reports of the existence in China and Japan of a bamboo with apparently four-angled stems have generally been regarded as myth, or at least as due to some disease or abnormal condition of a species whose stems, when properly developed, have a round cross-section. Of the existence of such a bamboo there cannot, however, now be any kind of doubt*" (McGowan, 1885, 1889; Thiselton Dyer, 1885).

In 1818 Gabriel Lamé (1795-1860) published a small booklet which generalized the circle to include square and rectangles, rhombus, and supercircles, and spheres into cubes and beams, ellipsoids and ovaloid, all captured into one single equation (Lamé, 1818). In 1997 Lamé curves and surfaces were generalized to any symmetry, as the Superformula (*A Generic Geometric Transformation That Unifies A Wide Range Of Natural And Abstract Shapes*, Gielis, 2003; Fig. 1).

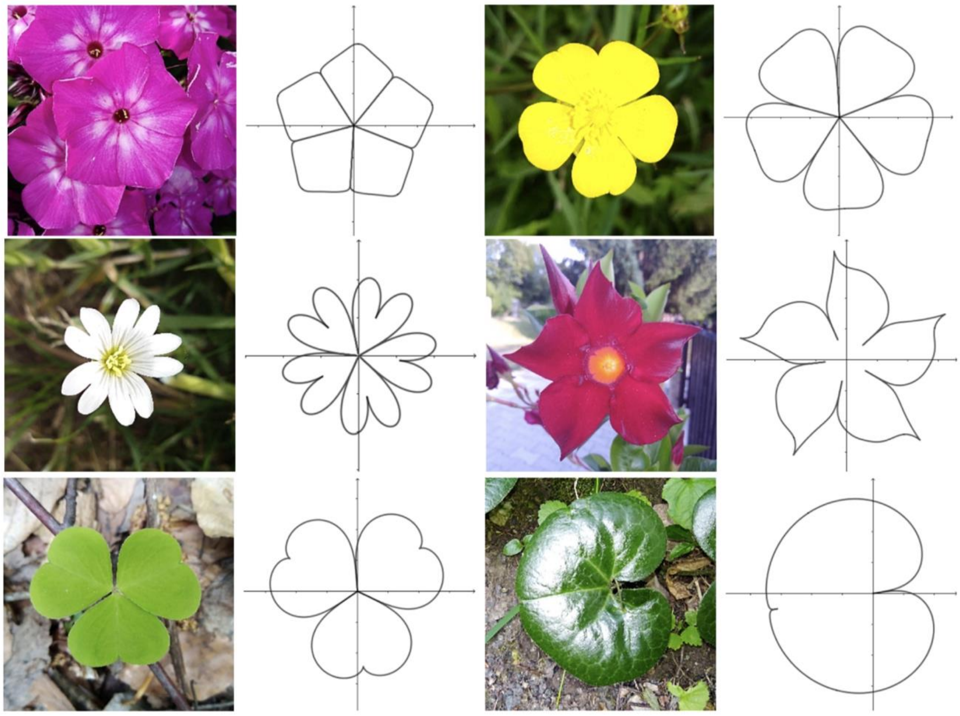

Fig. 1. Different leaves and flowers with the Superformula (Spichal, 2020)

The Superformula describes abstract shapes as different as triangles, polygons, prisms and cubes and natural shapes like diatoms, starfish, flowers and molluscs (Fig. 1). The names *superformula* and *supershapes* originate from the original connection to supercircles, superellipses and superquadrics, another name for Lamé curves and surfaces. Mathematicians changed the name into *Gielis curves, surfaces, (sub-) manifolds* and *transformations* or *Gielis Formula* (Gielis et al. 2005; Morales & Bobadilla, 2008; Koiso and Palmer, 2008; Matsuura, 2015).

2. Materials and methods

2.1. Gielis Equation and Simplified Gielis Equations

Gielis curves are a generalization of Lamé curves, and hence of the circle and the Pythagorean theorem, defined by (Gielis, 2003, 2017):

$$\varrho(\vartheta; A, B, n_1, n_2, n_3) = \frac{1}{\sqrt[n_1]{\left|\frac{1}{A}\cos\left(\frac{m}{4}\vartheta\right)\right|^{n_2} + \left|\frac{1}{B}\sin\left(\frac{m}{4}\vartheta\right)\right|^{n_3}}} \cdot f(\vartheta) \qquad (1)$$

This generalization of circles and Lamé curves ($m = 4$ and $n_{1,2,3} = n$; $f(\vartheta) = R$) result from using polar coordinates and generalizing the symmetry beyond that of the square. This equation can be understood as a transform to any planar curve $f(\vartheta)$ and can easily be extended to three and more dimensions.

Equation 2 is the Simplified Gielis Equation SGE, a low parameter version of Equation 1 with m = 1 and n_2 = n_3 = 1:

$$\varrho(\vartheta; m = 1, n_1, n_2 = n_3 = 1) = \frac{1}{\sqrt[n_1]{\left|\cos\left(\frac{\vartheta}{4}\right)\right| + \left|\sin\left(\frac{\vartheta}{4}\right)\right|}} \quad (2)$$

2.2. Measurements and data acquisition

All methods and data acquisition methodologies are described in the cited literature. For leaves, the relevant measures are indicated in Fig. 2 (Shi et al. 2015):

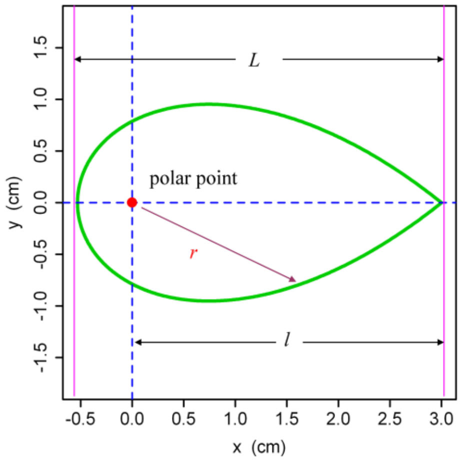

Fig. 2: Leaf shape and size parameters n, L, l are related as $L = (1 + 2^{-1/2n}).l$

2.3. Biogeom: An R-Package

All methodologies of scanning and processing natural shapes are detailed in the relevant literature and in an R-package called 'biogeom' that simulates and fits many shapes founding nature (Shi et al. 2022). The package incorporates novel universal parametric equations that generate the profiles of bird eggs, flowers, linear and lanceolate leaves, seeds, starfish, and tree-rings, and three growth-rate equations that generate the profiles of ovate leaves and the ontogenetic growth curves of animals and plants. 'biogeom' includes several empirical datasets comprising the boundary coordinates of bird eggs, fruits, lanceolate and ovate leaves, tree rings, seeds, and sea stars. In addition, the package includes sigmoid curves derived from the three growth-rate equations, which can be used to model animal and plant growth trajectories and predict the times associated with maximum growth rate.

3. Results

3.1. Square Bamboos, bamboo culms and bamboo meristems

The model was tested on 30 adult culms of *Chimonobambusa utilis* (Huang et al. 2020). For 750 sections, both the outer and inner shape can be modelled with arbitrary accuracy using only a few numbers. A single equation with two parameters and one additional transformation parameter fits all 1436 inner and outer rings tested (Figure 3). The same methods were used to fit cross-sections of bamboo shoots (Wei et al. 2017; Fig. 4) and meristems (Wei and Shi, 2017; Figure5).

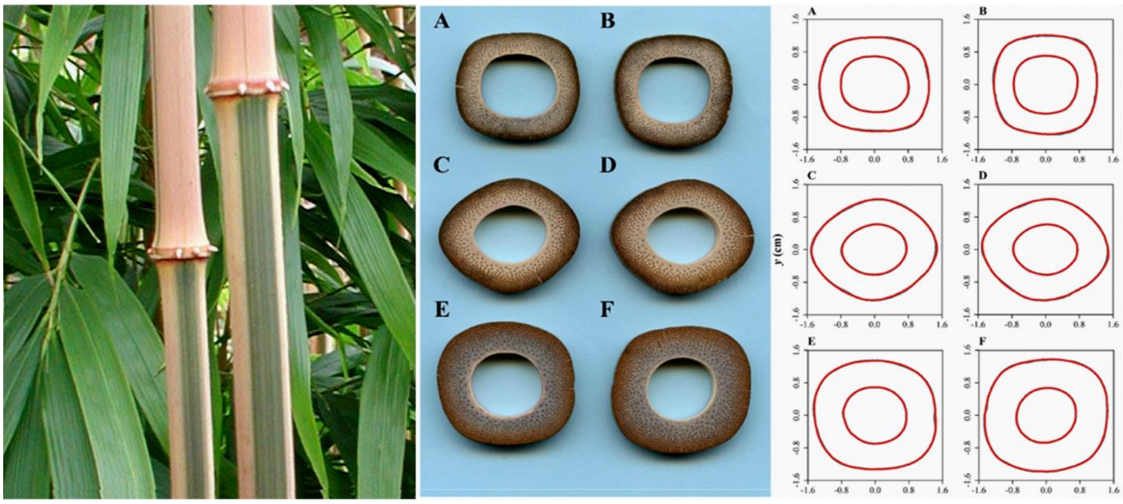

Fig. 3. Left: Culms of a variegated form of *Chimonobambusa quadrangularis*. Centre and right: Fitted curves for the six actual cross-sectional examples of *Chimonobambusa utilis* (centre) using the superellipse equation with a deformation parameter. The grey curves are the actual outer and inner rings; the red curves are fitted curves (Huang et al. 2020)

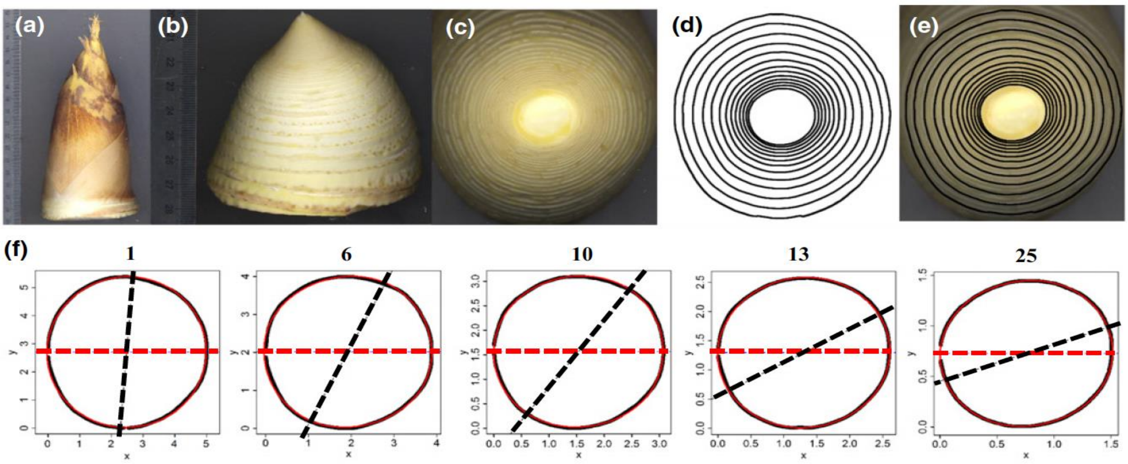

Fig. 4: Cross sections of bamboo shoots and their fits (Wei *et al*, 2017)

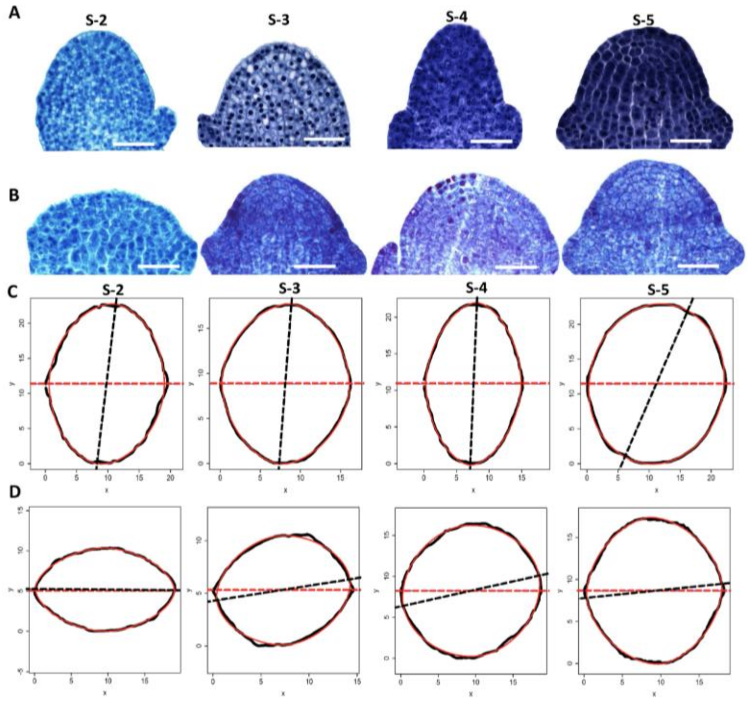

Fig. 5: Shoot apical meristems of thick walled Moso (A) and corresponding wild-type (B) at different development stages (Wei and Shi, 2017)

3.2. Forty-six species, more than thousand leaves, one equation.

The variation in leaf shape parameters for 46 species (Lin et al. 2016; Table 1) is shown in Fig. 6. Although there were significant differences in leaf shape parameters between species from different genera or from the same genus, the calculated leaf shape parameters for these 46 species only ranged from 0.02 to 0.1.

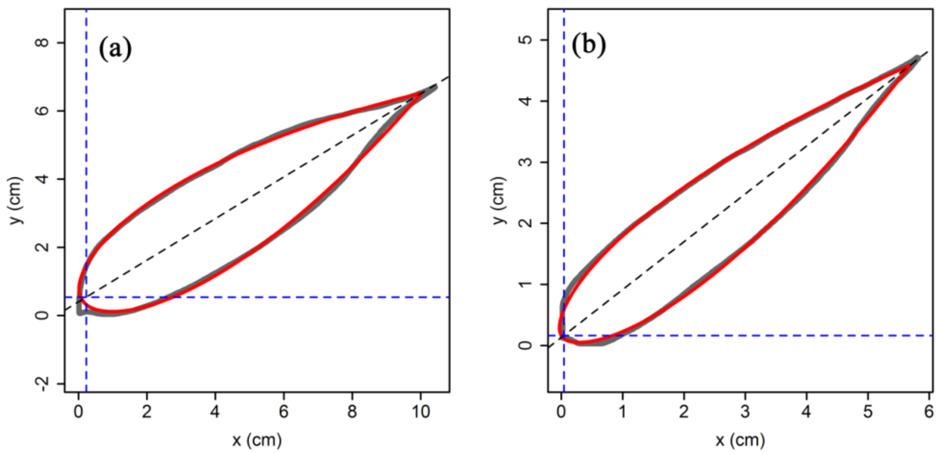

Fig. 6: Comparison between scanned leaf profile and predicted leaf profile from the simplified Gielis equation for (a) *Indosasa shibataeoides*; (b) *Phyllostachys bissetii*.

Table 1: 46 species of bamboo

Code	Latin name	Genus
1	*Bambusa emeiensis* var. *viridiflavus* Hsuen et Yi	*Bambusa*
2	*Bambusa multiplex* (Loureiro) Raeuschel ex Schultes & J. H. Schultes	*Bambusa*
3	*Bambusa multiplex* f.*fernleaf* (R. A. Young) T. P. Yi,	*Bambusa*
4	*Bambusa multiplex* var. *riviereorum*Maire	*Bambusa*
5	*Chimono bambusa marmorea* f. *variegata* (Mitford) Makino	*Chimonobambusa*
6	*Chimonobambusa neopurpure*a Yi	*Chimonobambusa*
7	*Chimonobambusa quadrangularis* (Franceschi) Makino	*Chimonobambusa*
8	*Chimono bambusa sichuanensis*(T. P. Yi) T. H. Wen	*Chimonobambusa*
9	*Chimono bambusa tumidissinoda* Hsueh & YI, ex. Ohrnb.	*Chimonobambusa*
10	*Indosasa shibataeoides* McClure	*Indosasa*
11	*Oligostachyumsulcatum* Z. P. Wang & G. H. Ye	*Oligostachyum*
12	*Phyllostachys arcana* cv. luteosulcata McClure,	*Phyllostachys*
13	*Phyllostachys aurea* Carrière ex Rivière & C. Rivière	*Phyllostachys*
14	*Phyllostachys aureosulcata* McClure	*Phyllostachys*
15	*Phyllostachys aureosulcata* f. pekinensis J.L. Lu	*Phyllostachys*
16	*Phyllostachys aureosulcata* f. spectabilis C.D. Chu & C.S. Chao	*Phyllostachys*
17	*Phyllostachys bissetii* McClure	*Phyllostachys*
18	*Phyllostachys dulcis* McClure	*Phyllostachys*
19	*Phyllostachys edulis* (Carrière) J. Houzeau	*Phyllostachys*
20	*Phyllostachys glauca* McClure	*Phyllostachys*
21	*Phyllostachys heteroclada* Oliver	*Phyllostachys*
22	*Phyllostachys edulis 'Gracilis'* (W.Y. Hsiung C.S. Chao) & S. Renvoize	*Phyllostachys*
23	*Phyllostachys nidularia* Munro	*Phyllostachys*
24	*Phyllostachys nigra* f.*henonis* (Mitford) Muroi	*Phyllostachys*
25	*Phyllostachys nigra* (Loddiges ex Lindley) Munro	*Phyllostachys*
26	*Phyllostachys sulphurea* var.*viridis* (Carrière) Rivière & C. Rivière	*Phyllostachys*
27	*Phyllostachys violascens* (Carrière) Rivière & C. Rivière	*Phyllostachys*

28	*Pleioblastus argenteostriatus* (Regel) Nakai	*Pleioblastus*
29	*Pleioblastus chino* (Franchet & Savatier) Makino	*Pleioblastus*
30	*Pleioblastus distichus* (Mitford) Nakai	*Pleioblastus*
31	*Pleioblastus fortunei* (Van Houtte) Nakai	*Pleioblastus*
32	*Pleioblastus gramineus* f. *monstrispiralis* (Y. Okada) Muroi & H.Hamada	*Pleioblastus*
33	*Pleioblastus kongosanensis* f. *aureostriatus Muroi & Y. Tanake*	*Pleioblastus*
34	*Pleioblastus maculatus* (McClure)C. D. Chu & C. S. Chao	*Pleioblastus*
35	*Pleioblastus simonii*f. *heterophyllus* (Makino & Shirasawa) Muroi	*Pleioblastus*
36	*Pleioblastus yixingensis* S. L. Chen & S. Y. Chen	*Pleioblastus*
37	*Pseudosasa amabilis* var. *convexa* Z. P. Wang & G. H. Ye	*Pseudosasa*
38	*Pseudosasa japonica* var. tzutsumiana (Siebold & Zuccarini ex Steudel) Makino ex Nakai	*Pseudosasa*
39	*Semiarundinaria densiflora* (Rendle) T. H. Wen	*Semiarundinaria*
40	*Semiarundinaria sinica* T. H. Wen	*Semiarundinaria*
41	*Shibataea chinensis* Nakai	*Shibataea*
42	*Sinobambusa tootsik* (Makino) Makino	*Sinobambusa*
43	*Indocalamus pedalis* (Keng) P.C. Keng	*Indocalamus*
44	*Indocalamus pumilus* Q.H. Dai and C.F. Keng	*Indocalamus*
45	*Indocalamus barbatus* McClure	*Indocalamus*
46	*Indocalamus victorialis* P.C. Keng	*Indocalamus*

The lower values refer to shapes of more linear-lanceolate type, such as *Pleioblastus chino, Pleioblastus simonii* f. heterophyllus, *Pleioblastus gramineus* f. monstrispiralis, *Chimonobambusa tumidissinoda* and *Phyllostachys edulis*. The leaves described by the higher values of the shape parameter n are somewhat broader in shape, with *Shibataea chinensis, Indosasa shibaeatoides* and *Bambusa multiplex* var. riviereorum. The rest of the leaves are by and large in the range *n*=0.03 to 0.08. The leaf shape is only slightly variable in *Indocalamus* species (Fig. 7).

The length of the leaves of these 46 bamboo species follows the Weibull distribution, rather than the normal distribution. This distribution comes from the study of different particle sizes in crushed particles, sand or volcanic ash, or from failure rates in products, following power laws. This implies that the distribution of leaf length in bamboo follows a power law. The shape of all leaf bamboo species is very well described by the simplified Gielis equation. The shape of bamboo leaves is encoded in two parameters, one for shape *n* (Equation 2) and one for size and the predicted leaf shape matched the observed leaf shape perfectly for all bamboo species (Fig. 6). The goodness of fit shown also provides convincing evidence for the bilateral symmetry in bamboo leaves, with all coefficients of determination higher than 0.980; the regression line deviated only slightly from the straight-line $y = x$.

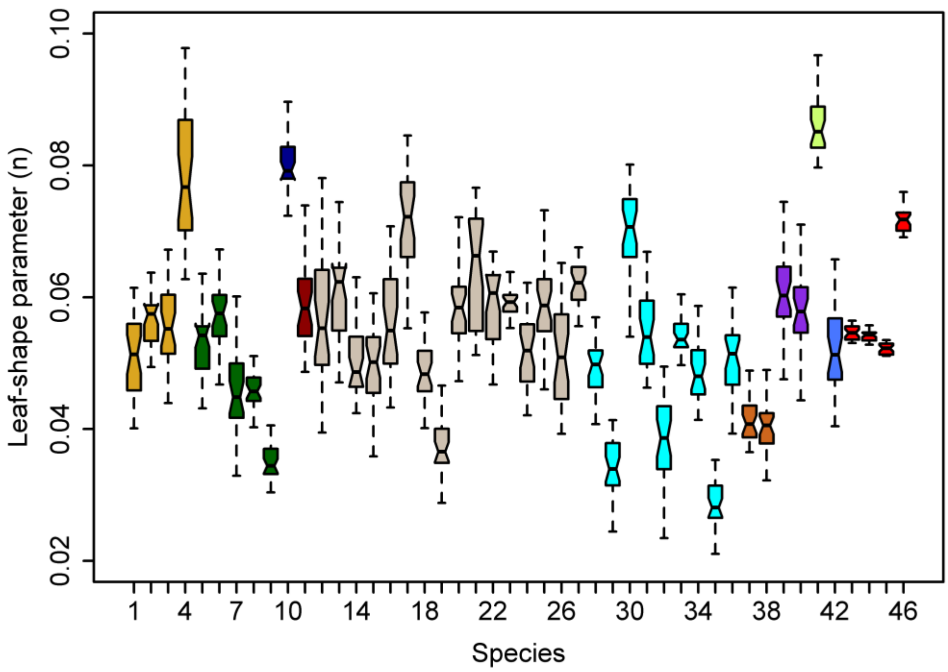

Fig. 7: Leaf shape parameter *n* for 46 species of bamboo. Colour codes are used for different genera; numbering in Table 1 (Lin et al. 2016).

4. Discussion

Modelling with the modified Gielis equation with a reduced number of parameters is clearly applicable to a wide range of temperate bamboo genera and species, and many other plants (Shi et al. 2018a). The range of shapes and sizes, from the small leaves of *Bambusa multiplex* var. riviereorum to the large leaves of *Indocalamus victorialis*, and from the linear-lanceolate leaves of *P. linearis* to the broader leaves of *Shibataea*, can be efficiently modelled with one equation, and only two parameters. Although the leaf sizes (length and width) of the 46 bamboo species in the study differ considerably, their leaf shapes vary relatively little, but the difference can be clearly identified and quantified. There are only two parameters, one for length and one for shape, which fully accounts for shape variations. In fact, even two different versions of the SGE can be used (Yao et al. 2022) and all data have been used in various applications to investigate allometric relationships (Shi et al. 2018b).

While in the past bamboo leaf blades were characterized by the quantitative measures of length and width with additional qualitative traits such as linear-lanceolate or oblong-lanceolate, we now have clear quantitative numerical values for a qualitative trait. Besides being used in identification of species, this can also be used as a very precise method in ecology, taxonomy and genetic diversity studies of bamboo without the need for molecular markers or any similar method for accurate identification. The shape, area and the variability of leaves of individual plants and populations can also be quantified (Lian et al. 2023). Every section of bamboo shoots, bamboo meristems or rhizomes can be quantified with any required precision and any of these quantifications can aid in the study of properties of any organ or the complete plants. The focus on bamboo leaves and culms in this contribution only represents a very small part of our research in plants.

Conflict of Interest

The authors declare there is no conflict of interest.

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